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2004 J. Phys. A: Math. Gen. 37 7397

(http://iopscience.iop.org/0305-4470/37/30/001)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 7397–7406 PII: S0305-4470(04)78079-8

An integrable model with a non-reducible three particle *R*-matrix

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Received 18 March 2004, in final form 3 June 2004
Published 14 July 2004
Online at stacks.iop.org/JPhysA/37/7397
doi:10.1088/0305-4470/37/30/001

Abstract

We define an integrable lattice model which, in the notation of Yang, in addition to the conventional two-particle *R*-matrices also contains non-reducible three-particle *R*-matrices. The corresponding modified Yang–Baxter equations are solved, and an expression for the transfer matrix is found as a normal-ordered exponential of a (non-local) Hamiltonian.

PACS numbers: 02.30.Ik, 05.50.+k, 11.15.Ha

1. Introduction

It is well known that integrable models have the following factorization property: any multi-particle scattering amplitude can be reduced to the product of two-particle scattering amplitudes. The independence of the order in which the successive two-particle scatterings take place leads to the Zamolodchikovs so-called triangle equations for the *S*-matrix [1]. In statistical physics, in the context of integrable lattice systems, the same property was analysed earlier by Yang [2] and Baxter [3] using the so-called *R*-matrices and the equations encountered were denoted Yang–Baxter equations (YBE) by L Faddeev [4].

From the *R*-matrices, one can construct the transfer matrices of the integrable lattice systems and the Yang–Baxter equations impose sufficient conditions on the *R*-matrices to ensure that transfer matrices $\tau(u)$ and $\tau(v)$ with different so-called spectral parameters u and v commute.

In the analysis of the Zamolodchikovs the factorization property was a consequence of an assumed Lorentz invariance in the model. Thus it is natural to ask if it is possible to construct an integrable lattice model with a commuting family of transfer matrices $\tau(u)$, where the transfer matrices are not products of only two-particle *R*-matrices, as is the case

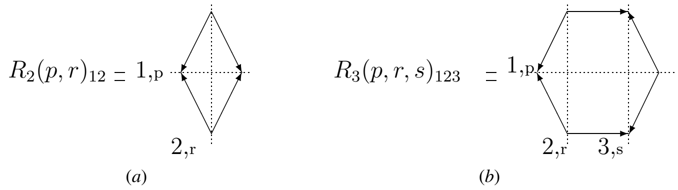

Figure 1. Graphical representations of two (*a*) and three (*b*) particle *R*-matrices.

for the presently known integrable models. An obvious, interesting question is then, if such lattice models can be associated with a Lorentz invariant continuum theory.

In this paper we will address this problem and will analyse the possibility of construction of an integrable model with three-particle *R*-matrices which cannot be reduced to products of three two-particle *R*-matrices. In a previous article [5], we have considered a model which appeared naturally as a simplification of the so-called sign-factor representation of the three-dimensional Ising model (3DIM) on a dual body centred cubic (DBBC) lattice[4]. A central point in the construction is the appearance of a two-dimensional *random* Manhattan lattice (ML). Such a Manhattan lattice appears also in the sign-factor model (SFM) of the 3DIM formulated on a cubic lattice [6], where the hopping of fermions from site to site is allowed only along the ML arrows. This directed hopping model can be described as a product of two-particle *R*-matrices. However, in the case where the surfaces were embedded in a DBCC lattice, we discovered the presence of three-particle *R*-matrices which cannot be reduced to the product of two particle *R*-matrices. Graphically they could be represented as honeycombs [9], see figure 1(*b*).

In this paper we go further and construct an integrable model where the transfer matrix is a product of two-particle R_2-matrices and (non-reducible) three-particle R_3-matrices. The model is defined in section 2. In section 3 we show that the corresponding Yang–Baxter equations ensure that transfer matrices with different spectral parameters commute and we present a nontrivial solution. We also obtain an explicit representation of the transfer matrix as the normal-ordered exponential of a (non-local) Hamiltonian.

2. Formulation of the model and definition of the *R*-matrices

In [6] the SFM for the 3DIM was defined as a theory of fermions interacting with an induced $Z(2)$ gauge field on 2D random surfaces embedded in a three-dimensional regular cubic lattice. Fermions were hopping along the directed links of a certain random ML. In [5], we generalized this construction of a SFM to the case of a 3DIM on a DBCC lattice and encountered, as mentioned above, not only the ordinary two-particle *R*-matrices but also three-particle *R*-matrices which could not be factorized into the product of three two-particle *R*-matrices. We initiated the analysis of the SFM by considering the restriction of the model to a certain regular ML lattice and investigated its integrability.

In this paper we study in the same spirit a model on a more complicated regular ML. Also this model has three-particle R_3- and ordinary two-particle R_2-matrices, but due to the different structures of the ML their appearance in the transfer matrix is different and we will show that

[4] This representation of the Ising model is in many respects the natural generalization of the representation of the two-dimensional Ising model as a sum over random walks. Here it becomes a sum over certain random surfaces, which can be related to fermionic strings.

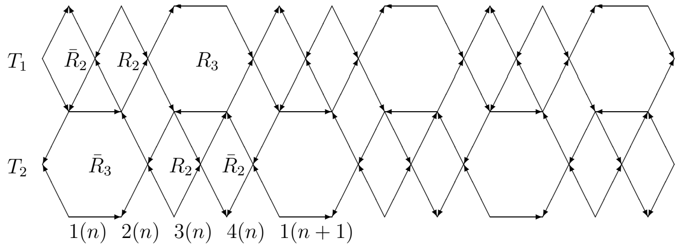

Figure 2. The transfer matrix of the model.

the corresponding YBEs, unlike the case considered in [5], have spectral parameter-dependent solutions.

The transfer matrix of the model is constructed as a product of two monodromy matrices T_1 and T_2:

$$\tau = \mathrm{tr}\, T_1 T_2 \tag{1}$$

where the matrices T_i are defined as shown in figure 2 in terms of the R-matrices

$$T_1 = \prod_n R_3(n) R_2(n) \bar{R}_2(n) \qquad T_2 = \prod_n \bar{R}_2(n) R_2(n) \bar{R}_3(n). \tag{2}$$

The T-matrices are acting on isomorphic quantum spaces which are a product of two-dimensional spin-spaces at the horizontal sites $1(n), 2(n), 3(n), 4(n), \ldots$ in figure 2, and they also act on isomorphic auxiliary spaces constructed from the left and right boundary sites in figure 2. The trace in (1) is taken with respect to the auxiliary spaces.

Following the technique of [5] one can express graphically the R_2- and R_3-matrices as shown in figure 1 and the transfer matrix τ as shown in figure 2. Alternatively this model can be formulated as a 2D quantum field model on a ML which can be obtained by repeating the two rows in figure 2 in a vertical direction. This lattice is invariant under translations of the block of two R_3 and four R_2 matrices in horizontal direction as well as in time direction. Later we will show that the partition function of the model, $Z = \mathrm{tr}(\tau)^{N_0}$, can be represented as a functional integral over scalar fermions as

$$Z = \mathrm{tr}(\tau)^{N_0} = \int D\bar{\psi}\, D\psi\, \mathrm{e}^{\bar{\psi}\mathcal{A}\psi + \psi\bar{\psi}} \tag{3}$$

where the matrix A_{ij} defines the hopping parameters along arrows on the corresponding ML. The matrix A inherits the translational invariance of the associated ML. The equivalence between the formulation in terms of scalar fermions as alluded to on the rhs of equation (3) and the formulation in terms of the R-matrices and τ are demonstrated by passing to a coherent fermionic state basis [8] and using the technique developed in [7, 10] and will be given below.

Let us recall the definition of the R-matrices. The two-particle $(R_2)_{ij}$-matrix is an operator acting on the direct product of two two-dimensional spaces V_i and V_j with basis elements $|i_1\rangle$ and $|j_1\rangle$, respectively, as

$$(R_2)_{ij}|i_1\rangle \otimes |j_1\rangle = (-1)^{p(i_2)(p(j_1)+p(j_2))} (R_{ij})_{i_1 j_1}^{i_2 j_2} |i_2\rangle \otimes |j_2\rangle \tag{4}$$

and can be represented graphically as shown in figure 1(a).

Similarly, the three-particle $(R_3)_{ijk}$-matrix is acting on the direct product of three two-dimensional spaces V_i , V_j and V_k with basis elements $|i_1\rangle$, $|j_1\rangle$ and $|k_1\rangle$

$$(R_3)_{ijk}|i_1\rangle \otimes |j_1\rangle \otimes |k_1\rangle = (-1)^{(p(i_2)+p(j_2))(p(k_1)+p(k_2))+p(i_2)p(j_2)} (R_{ijk})_{i_1 j_1 k_1}^{i_2 j_2 k_2} |i_2\rangle \otimes |j_2\rangle \otimes |k_2\rangle \tag{5}$$

and can be represented graphically as in figure 1(*b*). In the expressions (4) and (5) the parity factor $p(\alpha)$ takes into account the graded character of the elements $|\alpha\rangle$ of the Fock space of scalar fermions

$$p(\alpha) = n \qquad |\alpha\rangle = \prod_{i=1}^{n} c_i^{+}|0\rangle. \tag{6}$$

The basic states of the vector spaces V_I on each site i are $|k\rangle, k = 0, 1$ with $c|0\rangle = 0, |1\rangle = c^{+}|0\rangle$.

In order to have a simple hopping model on the ML for both R_2 and R_3, we follow the construction of corresponding matrices in the SFM of the 3DIM on the DBBC lattice [5] and use as an ansatz an exponential of a quadratic form (see (3)) of fermionic creation–annihilation operators

$$R_l =: \exp c_i^{+} A_{ij}^{(l)} c_j : \qquad l = 2, 3 \tag{7}$$

where the notion : $(\cdot)$: means normal ordering for the odd i, j and anti-normal(hole) ordering for the even i, j. The matrix elements $\left(A_{ij}^{(l)} - \delta_{ij}\right), l = 2, 3$ coincide up to a sign with the hopping parameters $\mathcal{A}_{ij}$ in (3)

$$\mathcal{A}_{ij} = \begin{matrix} A_{ij}^{(l)} - \delta_{ij} & i \text{ odd} & j \text{ even, odd} \\ \delta_{ij} - A_{ij}^{(l)} & i \text{ even} & j \text{ even, odd.} \end{matrix} \tag{8}$$

It is straightforward to calculate the matrix elements of R_l from (7) in the Fock space basic. In particular we note that the fermion number is conserved

$$\begin{matrix} (R_3)_{ijk}^{\bar{i}\bar{j}\bar{k}} \neq 0 & \text{if} & i+j+k = \bar{i}+\bar{j}+\bar{k} & i, \bar{i}, \ldots = 0, 1 \\ (R_2)_{ij}^{\bar{i}\bar{j}} \neq 0 & \text{if} & i+j = \bar{i}+\bar{j} & i, \bar{i}, \ldots = 0, 1 \end{matrix} \tag{9}$$

$$R_3 = \begin{pmatrix} R_{000}^{000} & & & & & & & \\ & R_{001}^{001} & R_{010}^{001} & & R_{100}^{001} & & & \\ & R_{001}^{010} & R_{010}^{010} & & R_{100}^{010} & & & \\ & & & R_{011}^{011} & & R_{101}^{011} & R_{110}^{011} & \\ & R_{001}^{100} & R_{010}^{100} & & R_{100}^{100} & & & \\ & & & R_{011}^{101} & & R_{101}^{101} & R_{110}^{101} & \\ & & & R_{011}^{110} & & R_{101}^{110} & R_{110}^{110} & \\ & & & & & & & R_{111}^{111} \end{pmatrix} \tag{10}$$

$$R_2 = \begin{pmatrix} R_{00}^{00} & & & \\ & R_{01}^{01} & R_{01}^{01} & \\ & R_{01}^{10} & R_{10}^{10} & \\ & & & R_{11}^{11} \end{pmatrix}. \tag{11}$$

The matrix elements $(R^{(3)})_{ijk}^{\bar{i}\bar{j}\bar{k}}$ in the expression (10) are connected with the A_{ij}s in (7) by the following equations:

$$\begin{matrix} R_{000}^{000} = \left(1 + A_{11}^{(3)}\right)\left(1 + A_{33}^{(3)}\right) - A_{13}^{(3)} A_{31}^{(3)} & R_{101}^{101} = 1 \\ R_{011}^{011} = \left(1 + A_{11}^{(3)}\right)\left(1 - A_{22}^{(3)}\right) + A_{12}^{(3)} A_{21}^{(3)} & R_{001}^{001} = 1 + A_{11}^{(3)} \\ R_{110}^{110} = \left(1 + A_{33}^{(3)}\right)\left(1 - A_{22}^{(3)}\right) + A_{23}^{(3)} A_{32}^{(3)} & R_{111}^{111} = 1 - A_{22}^{(3)} \\ R_{010}^{010} = -\det(A^{(3)} - 1) & R_{100}^{100} = 1 + A_{33}^{(3)} \end{matrix}$$

$$
\begin{aligned}
R_{001}^{010} &= A_{23}^{(3)}\left(1 + A_{11}^{(3)}\right) - A_{13}^{(3)}A_{21}^{(3)} & R_{010}^{001} &= A_{32}^{(3)}\left(1 + A_{11}^{(3)}\right) - A_{31}^{(3)}A_{12}^{(3)} \\
R_{010}^{100} &= A_{12}^{(3)}\left(1 + A_{33}^{(3)}\right) - A_{32}^{(3)}A_{13}^{(3)} & R_{100}^{010} &= A_{21}^{(3)}\left(1 + A_{33}^{(3)}\right) - A_{23}^{(3)}A_{31}^{(3)} \\
R_{011}^{110} &= A_{13}^{(3)}\left(1 - A_{22}^{(3)}\right) + A_{12}^{(3)}A_{23}^{(3)} & R_{110}^{011} &= A_{31}^{(3)}\left(1 - A_{22}^{(3)}\right) + A_{21}^{(3)}A_{32}^{(3)}. \\
R_{001}^{100} &= -A_{13}^{(3)} & R_{100}^{001} &= -A_{31}^{(3)} \\
R_{101}^{110} &= A_{23}^{(3)} & R_{110}^{101} &= A_{32}^{(3)} \\
R_{011}^{101} &= A_{12}^{(3)} & R_{101}^{011} &= A_{21}^{(3)}.
\end{aligned}
\tag{12}
$$

The two-particle R_2-matrix elements can be obtained from these expressions by taking $A_{i3}^{(3)} = A_{3j}^{(3)} = 0$ everywhere.

There are also some additional model-dependent constraints for the R_3 matrix para-meters coming from the specific lattice used

$$A_{22}^{(3)} - 1 = A_{13}^{(3)} = A_{31}^{(3)} = 0, \tag{13}$$

These conditions express algebraically that we (by the lattice construction) have no hoppings across the hexagon (see figure 2), and this is the reason the R_3-matrix cannot be reduced to the product of three R_2s, as one might have expected from general factorization properties. Going further back to the original full SFM on a random ML coming from the 3DIM, the hexagon R_3-matrices are precisely associated with a part of the embedded surfaces which carry a curvature, unlike the square R_2 matrices which are associated with flat parts of the embedded surfaces. In the piecewise linear geometry it is well known that the curvature

$$K_n = \frac{\pi}{4}(4 - n) \tag{14}$$

is associated with the n-faces of two-dimensional complexes. Therefore $K_6 = -\pi/2$, while $K_4 = 0$.

Some remark is necessary to make here. Though the fermions in the expression (7) of R_3 matrix appeared quadratically, nevertheless the model under consideration is not a model of free particles due to presence of curvature. They effectively are interacting via gravitational background.

Viewing the R-matrices as associated with particle scattering one attaches, using the language of integrable systems, a spectral parameter to each of the particles, the spectral parameters being connected to the rapidities of the particles. Therefore, one expects that the R_3-matrix in general depends on three spectral parameters (p, r, s), while R_2 depends on two spectral parameters (p, r).

3. Yang–Baxter equations and their solutions

The integrability conditions for the model can be found in the standard way by defining and solving the associated YBEs equations. One can obtain the local YBEs using the graphical representation shown in figure 3(a) and (b) as described, for instance, in [4]

$$\overrightarrow{R}_{12}(p,q)R_{234}(p,r,s)R_{12}(q,r)\bar{R}_{34}(q,s) = \bar{R}_{234}(q,r,s)\bar{R}_{23}(p,r)R_{34}(p,s)\,\overrightarrow{R}_{34}(p,q) \tag{15}$$

$$\overrightarrow{R}_{12}(p,q)\bar{R}_{23}(p,r)R_{34}(p,s)\bar{R}_{123}(q,r,s) = R_{12}(q,r)\bar{R}_{34}(q,s)R_{234}(p,r,s)\overleftarrow{R}_{34}(p,q). \tag{16}$$

These equations ensure that the transfer matrices $\tau(p,r,s) = \mathrm{Tr}(T_1T_2)$ (T_1 and T_2 are defined by (2)) with different spectral parameters commute

$$[\tau(p,r,s), \tau(q,r,s)] = 0. \tag{17}$$

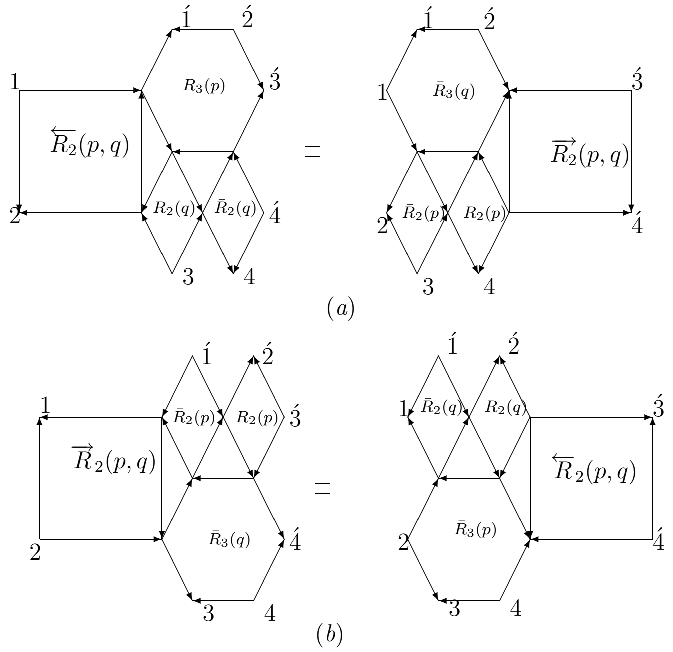

Figure 3. Yang–Baxter equations.

(We have just written the commutativity condition for one of spectral parameters.) This is what we mean by the system being integrable since we can use (17) to define an infinite set of mutually commuting, conserved charges $H_n(r,s)$ by expanding the transfer matrix $\tau(p,r,s)$ in powers of p

$$H_n(r,s) = \left.\frac{\mathrm{d}^n T(p,r,s)}{\mathrm{d}p^n}\right|_{p=p_0}. \tag{18}$$

We now look for nontrivial solutions to (15) and (16) where the intertwiner matrices $\overleftarrow{R}_{12}(p,q)$, $\overrightarrow{R}_{34}(p,q)$ have the structure (7) and (12). For convenience let us use a specific notation for the matrix elements of the R_2- and $\bar{R}_2$-matrices, while keeping $A^{(3)}_{ij}$s for the parametrization of R_3, $\bar{R}_3$-matrices

$$\begin{aligned}
&R^{00}_{00}(p,r) = a_1(p,r) && R^{11}_{11}(p,r) = a_2(p,r)\\
&R^{10}_{01}(p,r) = b_1(p,r) && R^{01}_{10}(p,r) = -b_2(p,r)\\
&\bar{R}^{00}_{00}(p,s) = \bar{a}_1(p,s) && \bar{R}^{11}_{11}(p,s) = \bar{a}_2(p,s)\\
&\bar{R}^{10}_{01}(p,s) = \bar{b}_1(p,s) && \bar{R}^{01}_{10}(p,s) = -\bar{b}_2(p,s).
\end{aligned} \tag{19}$$

With this notation, the YBEs (15) and (16) reduce to the following constraints on the parameters $a_k, \bar{a}_k, b_k, \bar{b}_k,\ k = 1, 2$:

$$\begin{aligned}
&\frac{a_1(p,r)a_2(p,r)}{b_1(p,r)b_2(p,r)} = f(r) && \frac{\bar{a}_1(p,r)\bar{a}_2(p,r)}{\bar{b}_1(p,r)\bar{b}_2(p,r)} = \bar{f}(s)\\
&a_1(p,r)\bar{b}_2(p,s) = \alpha_{12}(r,s) && a_2(p,r)\bar{b}_2(p,s) = \alpha_{21}(r,s)\\
&\bar{a}_1(p,s)b_2(p,r) = \bar{\alpha}_{12}(r,s) && \bar{a}_2(p,s)b_1(p,r) = \bar{\alpha}_{21}(r,s)
\end{aligned} \tag{20}$$

and the R_3 and $\bar{R}_3$ elements are connected with a_i, b_i by the relations

$$
\begin{array}{lll}
A^{(3)}_{23}(q,r,s) = y(r,s) & A^{(3)}_{21}(q,r,s) = x(r,s)\bar{a}_1(p,s) & A^{(3)}_{11}(q,r,s) = v(r,s)a_2(p,r) - 1 \\
\bar{A}^{(3)}_{23}(p,r,s) = \bar{y}(r,s) & \bar{A}^{(3)}_{21}(p,r,s) = \bar{x}(r,s)a_1(p,r) & \bar{A}^{(3)}_{11}(p,r,s) = \bar{v}(r,s)\bar{a}_2(p,s) - 1 \\
A^{(3)}_{12}(q,r,s) = w(r,s) & A^{(3)}_{32}(q,r,s) = u(r,s)\bar{b}_1(p,s) & A^{(3)}_{33}(q,r,s) = z(r,s)b_2(p,r) - 1 \\
\bar{A}^{(3)}_{12}(p,r,s) = \bar{w}(r,s) & \bar{A}^{(3)}_{32}(p,r,s) = \bar{u}(r,s)b_1(p,r) & \bar{A}^{(3)}_{33}(p,r,s) = \bar{z}(r,s)\bar{b}_2(p,s) - 1.
\end{array}
\tag{21}
$$

Here $f(r)$, $\bar{f}(r)$, $\alpha_{ij}(r,s)$, $\bar{\alpha}_{ij}(r,s)$, $y(r,s)$, $\bar{y}(r,s)$, $w(r,s)$, $\bar{w}(r,s)$, $x(r,s)$, $\bar{x}(r,s)$, $u(r,s)$, $\bar{u}(r,s)$, $v(r,s)$, $\bar{v}(r,s)$, $z(r,s)$, $\bar{z}(r,s)$, are arbitrary functions of variables r, s.

4. The transfer matrix

In order to calculate the transfer matrix of our model explicitly, expressed as a normal-ordered form of an exponential operator, one inserts (7), given by (19)–(21), into (2) (see figure 2). From figure 2 it follows that the double-row transfer matrix (2) is invariant under translations by four lattice spacings: we can restore the whole lattice structure by translation of the block of two R_3- and four R_2-matrices, either horizontally or vertically. After some algebra, using Wick's contraction theorem, we obtain

$$\tau(p,r,s) = \mathrm{tr}(T_1T_2) = F(r,s) : \exp H(p,r,s) : . \tag{22}$$

The number-valued pre-factor $F(r,s)$ of the normal-ordered exponential is

$$F(r,s) = \lambda_1^N + \lambda_2^N - \varepsilon_1^N - \varepsilon_2^N \tag{23}$$

where N is the number of constituent horizontal blocks in the chain and

$$
\begin{aligned}
\lambda_{1,2} &= \frac{\vartheta \pm \sqrt{\vartheta^2 - 4\varepsilon_1\varepsilon_2}}{2} \\
\vartheta &= (1-\bar{\alpha}_{12}\bar{\omega})(1-\bar{\alpha}_{21}\bar{y}) + (1-u\bar{z}\alpha_{12}\alpha_{21})(1-v\bar{x}\alpha_{12}\alpha_{21}) - 1 \\
\varepsilon_1 &= \bar{\alpha}_{21}\alpha_{12}^2\bar{x}\bar{z}\omega, \qquad \varepsilon_2 = \bar{\alpha}_{12}\alpha_{21}^2 uv\bar{y}.
\end{aligned}
$$

The operator $H(p,r,s)$ in (23) is a quadratic form of fermionic creation and annihilation operators:

$$H(p,r,s) = \sum_{i,j} H_{ij}(n-m)c_i^+(n)c_j(m). \tag{24}$$

This will be derived below.

In general H is a nonlocal operator, $H_{ij}(n-m)$ being a polynomial function of $\lambda_{1,2}$ and $\varepsilon_{1,2}$ of degrees $i-j$, or $N-(i-j)$, respectively. But by the translational invariance of the monodromy matrices, we can Fourier transform the $H(p,r,s)$ operators and present them in a compact form as

$$
\begin{aligned}
H(p,r,s) &= \sum_{n,m;i,j}^{N;4} H_{ij}(n-m)c_i^+(n)c_j(m) = \sum_{i,j;k}^{4;N} \bar{H}_{i,j}(k)c_i^+(k)c_j(k) \\
c_i^+(k) &= \frac{1}{\sqrt{N}}\sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}2\pi\frac{kn}{N}}c_i^+(n), \qquad c_j(k) = \frac{1}{\sqrt{N}}\sum_{n=1}^{N} \mathrm{e}^{-\mathrm{i}2\pi\frac{kn}{N}}c_j(n) \\
\bar{H}_{i,j}(k) &= \frac{1}{N}\sum_{n=1}^{N} \mathrm{e}^{\mathrm{i}2\pi\frac{kn}{N}}H_{i,j}(n).
\end{aligned}
\tag{25}
$$

Let us now finally discuss how the calculation of the matrix elements of $\bar{H}_{i,j}(k)$ can be done by using the technique of fermionic coherent states with Grassmannian variables, as formulated in [8, 7, 11]. On this basis the product and the trace of the operators are obtained by integrating over the Grassmann variables. Choosing a anti-coherent and coherent basis for particles on the odd $(2k-1; n)$ and even $(2k; n)$ boundary sites of the monodromy matrices (2), respectively, (see figure 2), we define

$$\begin{aligned}
&|\psi_{2k-1}(n)\rangle = \mathrm{e}^{\psi_{2k-1}(n)c^+_{2k-1}(n)}|0\rangle && \langle\bar{\psi}_{2k-1}(n)| = \langle 0|\,\mathrm{e}^{c_{2k-1}(n)\bar{\psi}_{2k-1}(n)} \\
&|\bar{\psi}_{2k}(n)\rangle = \left(c^+_{2k}(n) - \bar{\psi}_{2k}(n)\right)|0\rangle && \langle\psi_{2k}(n)| = \langle 0|(c_{2k}(n) + \psi_{2k}(n)) \\
&\langle\bar{\psi}_{2k-1}(n)|\psi_{2k-1}(n)\rangle = \mathrm{e}^{\bar{\psi}_{2k-1}(n)\psi_{2k-1}(n)} && \langle\psi_{2k}(n)|\bar{\psi}_{2k}(n)\rangle = \mathrm{e}^{\psi_{2k}(n)\bar{\psi}_{2k}(n)}.
\end{aligned} \tag{26}$$

These states are by construction eigenstates of the fermionic creation and annihilation operators c^+_k and c_k with eigenvalues $\psi_k(n)$ and $\bar{\psi}_k(n)$

$$\begin{aligned}
&c_{2k} \mid \psi_{2k}(n)\rangle = -\psi_{2k}(n) \mid \psi_{2k}(n)\rangle && \langle\bar{\psi}_{2k}(n) \mid c^+_{2k} = -\langle\bar{\psi}_{2k}(n) \mid \bar{\psi}_{2k}(n) \\
&c^+_{2k+1} \mid \bar{\psi}_{2k+1}(n)\rangle = \bar{\psi}_{2k+1}(n) \mid \bar{\psi}_{2k+1}(n)\rangle && \langle\psi_{2k+1}(n) \mid c_{2k+1} = -\langle\psi_{2k+1}(n) \mid \psi_{2k+1}(n).
\end{aligned} \tag{27}$$

We also attach coherent states $\chi_{2k}(n)$, $\bar{\chi}_{2k}(n)$, $\chi_{2k+1}(n)$ and $\bar{\chi}_{2k+1}(n)$ to the intermediate sites between the two transfer matrices τ_1 and τ_2. Then the full two row-transfer matrix in the coherent states basis, expressed via the one-row transfer matrices τ_1 and τ_2, can be written as

$$\begin{aligned}
\tau(\bar{\psi}, \psi) = \int D\bar{\chi}\, D\chi\, \mathrm{e}^{-\sum\bar{\chi}\chi}\, &\tau_1(\bar{\psi}_{2k-1}(n), \psi_{2k}(n), \bar{\chi}_{2k}(n), \chi_{2k-1}(n)) \\
&\cdot \tau_2(\bar{\chi}_{2k-1}(n), \chi_{2k}(n), \bar{\psi}_{2k}(n), \psi_{2k-1}(n))
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
\tau_i &= \mathrm{tr}\, T_i \qquad i = 1, 2 \\
\tau_1(\bar{\psi}, \psi, \bar{\chi}, \chi) &= \prod_{n=1}^{N}\prod_{k=1}^{2}\langle\bar{\psi}_{2k-1}(n)|\langle\psi_{2k}(n)|\tau_1|\bar{\chi}_{2k}(n)\rangle|\chi_{2k-1}(n)\rangle \\
\tau_2(\bar{\chi}, \chi, \bar{\psi}, \psi) &= \prod_{n=1}^{N}\prod_{k=1}^{2}\langle\bar{\chi}_{2k-1}(n)|\langle\chi_{2k}(n)|\tau_2|\bar{\psi}_{2k}(n)\rangle|\psi_{2k-1}(n)\rangle.
\end{aligned} \tag{29}$$

In order to define matrix multiplication in the coherent space basis we simply insert between operators the identity operators

$$\begin{aligned}
&\int \mathrm{d}\bar{\chi}_i(n)\,\mathrm{d}\chi_i(n)|\chi_i(n)\rangle\langle\bar{\chi}_i(n)|\,\mathrm{e}^{-\bar{\chi}_i(n)\chi_i(n)} = 1 \\
&\int \mathrm{d}\bar{\chi}_i(n)\,\mathrm{d}\chi_i(n)|\bar{\chi}_i(n)\rangle\langle\chi_i(n)|\,\mathrm{e}^{-\bar{\chi}_i(n)\chi_i(n)} = 1
\end{aligned} \tag{30}$$

for coherent and anti-coherent states, respectively. These relations simply express the completeness of the coherent state basis.

Now it is straightforward to express the transfer matrix in the coherent states basis. By inserting expressions (30) for the intermediate coherent states χ between R-operators in 2 and by considering matrix elements between external quantum states ψ, we obtain

$$\begin{aligned}
\tau(\bar{\psi}, \psi) &= \int D\bar{\chi}\, D\chi\, \mathrm{e}^{-\sum\bar{\chi}\chi} \prod \mathbf{R}_i(n)(\bar{\psi}, \psi, \bar{\chi}, \chi) \\
&= \int D\bar{\chi}\, D\chi\, D\bar{\chi}\, D\chi\, \exp\left\{\sum_{n-m=0,1}(\bar{\chi}_i(n)\Delta_{ij}(n-m)\chi_j(m)\right.
\end{aligned}$$

$$+\bar{\psi}_i(n)\bar{\Delta}_{ij}(n-m)\psi_j(m)+\bar{\psi}_i(n)\bar{\bar{\Delta}}_{ij}(n-m)\chi_j(m)$$
$$+\bar{\chi}_i(n)\bar{\bar{\bar{\Delta}}}_{ij}(n-m)\psi_j(m))\Bigg\}. \tag{31}$$

In this equation the matrix elements of the operator $R_l(\bar{c}, c)$ from (7) are represented as $\mathbf{R}(\bar{\psi}, \psi) = e^{\sum \bar{\psi}\psi} R_l(\bar{\psi}, \psi)$ by use of the coherent states.

The matrix Δ represents vacuum fluctuations, and its determinant will appear in the final expression after integration over internal states χ. After Fourier transformation of the Grassmannian variables ψ, χ, the Fourier transformed $\Delta(k)$ of the matrix $\Delta(n)$ becomes a 10×10 matrix (10 is the number of the intermediate states in each repeating block, as one can see in figure 2):

$$\Delta(k) = \frac{1}{N}\sum_{n=1}^{N}\Delta(n) = \begin{pmatrix} -1 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 & a_1\,e^{2i\pi\frac{k}{N}} & 0 \\ 0 & -1 & \bar{a}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ w & 0 & -1 & v & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & \bar{b}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \bar{b}_2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{x} & -1 & 0 & \bar{y} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{a}_2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_1 & -1 & 0 & a_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{z} & -1 & 0 \\ u\,e^{-2i\pi\frac{k}{N}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{32}$$

The function $F(p, r, s)$ in equation (22) can be expressed via the determinant of Δ as

$$F(p, r, s) = \det \Delta_{i,j}(n, m) = \prod_k \det \Delta_{ij}(k) \tag{33}$$

while $H(p, r, s)$ is defined by the Fourier transform of the matrix

$$\bar{H}_{i,j}(k) = \bar{\Delta}_{ij}(k) + \bar{\bar{\Delta}}_{im}(k)(\Delta)^{-1}_{ml}(k)\bar{\bar{\bar{\Delta}}}_{lj}(k). \tag{34}$$

We will write down here $\bar{H}_{i,j}(k)$ only for a simple case

$$\begin{aligned} &\alpha_{12} = \alpha_{21} = \bar{\alpha}_{12} = \bar{\alpha}_{21} = \alpha \qquad && f = \bar{f} = 1 \\ &w = \bar{y} = \bar{w} = y && u = v = x = z = \bar{u} = \bar{v} = \bar{x} = \bar{z} \end{aligned} \tag{35}$$

and have found following matrix elements of the Hamiltonian:

$$\begin{aligned}
&\bar{H}_{11}\left(k, \frac{b_1}{a_1}\right) = -A\left(\frac{b_1}{a_1}\right)^2 \alpha^2 u^2\left[\alpha w\,e^{i\frac{2\pi k}{N}} - \alpha^2 u^2 + 1\right] - 1 \\
&\bar{H}_{11}\left(k, \frac{b_1}{a_1}\right) = \bar{H}_{22}\left(k, \frac{a_1}{b_1}\right) = \bar{H}_{33}\left(-k, \frac{b_1}{a_1}\right) = \bar{H}_{44}\left(-k, \frac{a_1}{b_1}\right) \\
&\bar{H}_{14} = \bar{H}_{41} = -Au\alpha(1 - \alpha^2 u^2 - w\alpha) \\
&\bar{H}_{12}(k) = \bar{H}_{43}(-k) = -A\alpha\left(1 + e^{-i\frac{2\pi k}{N}}\alpha^2 u^2 - w\alpha\right) \\
&\bar{H}_{23}(k) = \bar{H}_{32}(-k) = -Au\alpha\,e^{i\frac{2\pi k}{N}}(1 - \alpha^2 u^2 - w\alpha) \\
&\bar{H}_{21}(k) = \bar{H}_{34}(-k) = -Au^2\alpha^2 w\,e^{i\frac{2\pi k}{N}}\left(1 + \alpha^2 u^2\,e^{-i\frac{2\pi k}{N}} - w\alpha\right) + w \\
&\bar{H}_{24}\left(k, \frac{b_1}{a_1}\right) = u\alpha^2\bar{H}_{42}\left(-k, \frac{b_1}{a_1}\right) = -A\left(\frac{a_1}{b_1}\right)^2 u\alpha^2\left(e^{i\frac{2\pi k}{N}} + 1\right) \\
&\bar{H}_{31}\left(k, \frac{b_1}{a_1}\right) = u^2\alpha^2 w\bar{H}_{13}\left(-k, \frac{b_1}{a_1}\right) = -A\left(\frac{b_1}{a_1}\right)^2 u^2\alpha\left(e^{i\frac{2\pi k}{N}} + 1\right)
\end{aligned} \tag{36}$$

where

$$A^{-1} = (1 - w\alpha)^2 + (1 - u^2\alpha^2) - 1 - 2\cos(2\pi k/N)u^2 w\alpha^3. \qquad (37)$$

It should be mentioned that the dependence on the spectral parameter p in the expressions above comes from the fraction a_1/b_1. As seen we have a nonlocal model of hopping fermions.

It is technically possible to transform the normal-ordered form (22) of the transfer matrix $\tau(p, r, s)$ into the exponential of the Hamiltonian by use of Fourier transformed basis (23), which now will contain interaction terms. But we prefer the normal-ordered form for the transfer matrix, since its kernel operator has a Gaussian form in the coherent states basis and it is relatively easy to obtain the eigensystem in this case.

Finally we would like to make the following remark. In the article [12] Zamolodchikov has defined some S-matrix for scattering of straight strings and formulated the analogue of the YBEs for them, called the tetrahedron equations, in order to have an integrable model. In the vertex formulation this S-matrix [13] has three initial and three final indices precisely as the R_3-matrix in our construction. Contrary to our situation, where particle number conservation is ensured by equation (9), the nontrivial (to us known) solutions of the tetrahedron equations do not have particle number conservation, except of the one case represented in [14].

Acknowledgments

JA, ShK and AS were supported by an INTAS grant. JA also acknowledges support by the EU network on 'Discrete Random Geometry', grant HPRN-CT-1999-00161 as well as by *MaPhySto*, Network of Mathematical Physics and Stochastics, funded by a grant from Danish National Research Foundation.

References

[1] Zamolodchikov A and Zamolodchikov Al 1978 *Phys. Lett.* B **72** 481
[2] Yang C N 1967 *Phys. Rev. Lett.* **19** 1312
[3] Baxter R J 1972 *Ann. Phys.* **70** 193
[4] Faddeev L D 1980 *Sov. Sci. Rev.* C **1** 107
[5] Ambjorn J, Khachatryan Sh and Sedrakyan A 2002 *From Gauge Theories to Integrable Models* ed V Gurzadyan and A Sedrakyan (Singapore: World Scientific)
[6] Kavalov A and Sedrakyan A 1986 *Phys. Lett.* B **173** 449
Kavalov A and Sedrakyan A 1987 *Nucl. Phys.* B **285** 264
[7] Sedrakyan A 1999 *Nucl. Phys.* B **554** 514
[8] Berezin F 1965 *The Methods of Second Quantization* (Moscow: Nauka)
Faddeev L 1975 Introduction to functional methods *Les Houches* session 28 ed R Balian and J Zinn-Justin
[9] Khachatryan Sh and Sedrakyan A 2002 *Phys. Lett.* A **293** 173
[10] Ambjorn J, Karakhanyan D, Mirumyan M and Sedrakyan A 2001 *Nucl. Phys.* B **599** 547
[11] Sedrakyan A 2002 *Statistical Field Theories* ed A Cappelli and G Mussardo p 67
[12] Zamolodchikov A 1981 *Commun. Math. Phys.* **79** 489
[13] Bazhanov V V and Stroganov Yu G 1982 *Teor. Mat. Fiz.* **52** 105
Korepanov I G 1993 *Commun. Math. Phys.* **154** 85
[14] Bazhanov V V and Stroganov Yu G 1984 *Nucl. Phys.* B **230** 435